$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad \underline{\text{FToC 1}}$$

Let $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$.

So if $F(x)$ is any other antiderivative of $f(x)$, we know

$$F(x)=g(x)+C$$

for $a<x<b$. Since F and g are continuous on $[a,b]$, it follows also that $F(x)=g(x)+C$ on $[a,b]$.

Plug in $x=a$

$$F(a)=g(a)+C$$

$$=\underbrace{\cancel{\int_a^a f(t)\,dt}}_{0}+C$$

$$=C$$

Plug in $x=b$:

$$F(b) = g(b) + C$$
$$= \int_a^b f(t)\,dt + C$$

$$F(b) - F(a) = [g(b)+C] - [g(a)+C]$$
$$= \left(\int_a^b f(t)\,dt + \cancel{C}\right) - \left(\cancel{C}\right)$$
$$= \int_a^b f(t)\,dt$$

The Fundamental Theorem of Calculus (part II)

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

Ex: FToC 1

① $\frac{d}{dx}\int_3^x \sqrt{t^3-1}\,dt = \sqrt{x^3-1}$

$f'(g(x)) \cdot g'(x)$

② $\underbrace{\frac{d}{dx}\int_9^{x^3} \sin(t^2-9)\,dt}_{f(g(x))} = \sin((x^3)^2-9) \cdot 3x^2$

$f(x) = \int_9^x \sin(t^2-9)\,dt \qquad g(x) = x^3$

Ex: FToC 2 $\quad \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$

① $\int_2^8 x^2\,dx = \frac{1}{3}x^3\Big|_2^8 = \frac{1}{3}\cdot 8^3 - \frac{1}{3}2^3$

$= \frac{512}{3} - \frac{8}{3} = \frac{504}{3} = 168$

② $\int_0^\pi \sin(x)\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0)$

$= -(-1) - (-1) = 2$

③ $\int_1^3 (x^2+2x-4)\,dx = \left(\frac{1}{3}x^3+x^2-4x\right)\Big|_1^3$

$= (9+\overset{6}{9-12}) - (\overset{\frac{1}{3}-3}{\frac{1}{3}+1-4}) = \frac{26}{3}$

$\int_1^4 \ln x\,dx$, $\int_{-1}^1 e^{-x^2}\,dx$

possible, but not yet

impossible (in closed form)

"Indefinite integral" = "most general antiderivative"

Ex: If $f(x) = x^2 + \cos x$, the most general antiderivative is $F(x) = \frac{1}{3}x^3 + \sin x + C$, i.e.,

$$\underbrace{\int (x^2+\cos x)\,dx}_{\text{indefinite integral}} = \frac{1}{3}x^3 + \sin x + C$$

Properties of the Indefinite Integral

① $\int c f(x)\,dx = c\int f(x)\,dx$

② $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Table of Basic Integrals

$$\int k\,dx = kx + C$$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)}$$

Power Rule

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C \quad (b>0,\ b\neq 1)$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

How do we compute $\int \tan x \, dx$?